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Finite-Dimensional Perturbations of Bounded Self-adjoint Operators

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1. INTRODUCTION

This paper gives a procedure by which information about a bounded self-adjoint operator A can be used to obtain information about the eigenvalues and corresponding eigenspaces of a bounded finite-dimensional perturbation B of A. B is not assumed to be self-adjoint. Such perturbation problems may arise, e.g., when a self-adjoint differential operator is subjected to a change of boundary conditions [2, Appendix].

Weinstein considered in [7] the case when A is compact and self-adjoint. He constructed, for each $\lambda \neq 0$, a matrix whose nullity gives the geometric multiplicity of λ as an eigenvalue of B. The matrix takes one of two different forms, depending on whether λ is or is not an eigenvalue of A.

The present paper considers the case that A is bounded and self-adjoint and that λ is in the meromorphy domain of A. It is shown that the corresponding Weinstein matrix can then be interpreted as a conveniently simple type of perturbation matrix in the sense of Aronszajn and Brown [2].

This result serves a dual purpose. First, it relates Weinstein's results to the general theory of finite-dimensional perturbations developed in [2]. Second, and more important, it implies that the Weinstein matrix corresponding to λ can be used as described in [2] to obtain (at least theoretically) the complete structure of the generalized eigenspace of B corresponding to λ (including of course, the geometric multiplicity of λ). In addition, it enables us to answer the questions raised in [7] concerning the applicability of Aronszajn's rule [1, 5, 8].

2. PERTURBATION MATRICES AND ARONSZAJN'S RULE

Let A be a bounded self-adjoint operator on a complex infinite-dimensional Hilbert space X, and let B be a bounded finite-dimensional perturbation of A. Then B can be written in the form

$$B = A + \sum_{i=1}^{n} (\cdot, q_i)\, p_i\,, \tag{2.1}$$

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with $q_1, \ldots, q_n, p_1, \ldots, p_n$ in X and $q_1, \ldots, q_n$ linearly independent. Note that B is not assumed to be self-adjoint.

Let $\mathscr{M}$ be the meromorphy domain of A; i.e., the union of the resolvent set $\rho(A)$ of A with the set of isolated eigenvalues of A of finite multiplicity. Since A is self-adjoint, $\mathscr{M}$ is an open connected neighborhood of infinity in the complex plane (namely, the complement of the essential spectrum of A). And, since B is a bounded finite-dimensional perturbation of A, the meromorphy domain of B is also $\mathscr{M}$ [2, 3, 5].

For $\lambda \in \rho(A)$, Weinstein defines [7] the "small" matrix $W_1(\lambda)$ corresponding to the representation (2.1) to be the $n \times n$ matrix with components $w_{ij} = \delta_{ij} + ((A - \lambda)^{-1} p_j, q_i)$ $(i, j = 1, \ldots, n)$. On the other hand, the perturbation matrix $M(\lambda)$ corresponding to (2.1) is defined [2] to be the matrix representation (with respect to $q_1, \ldots, q_n$) of the operator $P(A - \lambda)^{-1}(B - \lambda)$ acting on S, where S is the subspace spanned by $q_1, \ldots, q_n$ and P is the orthogonal projection of X onto S.

Specifically, $M(\lambda) = \{m_{ij}(\lambda)\}$ $(i, j = 1, \ldots, n)$ where

$$(A - \lambda)^{-1}(B - \lambda) q_j = \sum_{i=1}^{n} q_i m_{ij}(\lambda) + d_j, \tag{2.2}$$

with $d_j \in S^{\perp}$ $(j = 1, \ldots, n)$. It follows from (2.1), however, that

$$(A - \lambda)^{-1}(B - \lambda) q_j = q_j + \sum_{i=1}^{n} (q_j, q_i)(A - \lambda)^{-1} p_i. \tag{2.3}$$

We equate the right-hand sides of (2.2) and (2.3), then take the scalar product with $q_1, \ldots, q_n$ to obtain

$$\sum_{i=1}^{n} (q_i, q_k) m_{ij}(\lambda) = (q_j, q_k) + \sum_{i=1}^{n} (q_j, q_i)((A - \lambda)^{-1} p_i, q_k) \qquad (j, k = 1, \ldots, n).$$

Thus

$$M(\lambda) = Q^{-1} W_1(\lambda) Q, \tag{2.4}$$

where the nonsingular matrix Q has components $q_{ij} = (q_j, q_i)$ $(i, j = 1, \ldots, n)$.

$W_1(\lambda)$ and $M(\lambda)$ are $n \times n$ matrix valued functions defined and analytic for $\lambda \in \rho(A)$, and $\det W_1(\lambda) = \det M(\lambda)$ is a complex valued function analytic in $\rho(A)$ and meromorphic in $\mathscr{M}$. Let $\lambda_0 \in \mathscr{M}$ and $\omega(\lambda_0)$ be the order of λ_0 as a zero or pole of $\det W_1(\lambda)$. Then [2, Theorem II.6.2] the following form of Aronszajn's rule holds:

$$\omega(\lambda_0) = m_B(\lambda_0) - m_A(\lambda_0), \tag{2.5}$$

where $m_A(\lambda_0)$ is the multiplicity of λ_0 as an eigenvalue of A and $m_B(\lambda_0)$ is the *algebraic* multiplicity of λ_0 as an eigenvalue of B.

In case B is self-adjoint (2.5) reduces to the more familiar form of Aronszajn's rule for self-adjoint operators [8]. In the more general case considered here, Aronszajn's rule enables us to find the algebraic multiplicity of λ_0 as an eigenvalue of B rather than the geometric multiplicity. This result settles the question raised in [7] concerning Aronszajn's rule.

3. Points in the Resolvent Set of A; Type Exponents of the Perturbation Matrix

Before stating the next result we need to recall the definition of type exponents of a matrix given in [2].

Let $M(\lambda)$ be any $n \times n$ matrix whose components are analytic functions of λ in a neighborhood of the complex number λ_0. For $i = 1,..., n$ let $d_i(\lambda_0)$ be the largest real number such that $(\lambda - \lambda_0)^{d_i(\lambda_0)}$ is a factor of each of the $i \times i$ minor determinants of $M(\lambda)$. Then clearly

$$0 \leqslant d_1(\lambda_0) \leqslant d_2(\lambda_0) \leqslant \cdots \leqslant d_n(\lambda_0),$$

where $d_i(\lambda_0) = \infty$ if and only if all the $i \times i$ minor determinants of $M(\lambda)$ are identically zero in a neighborhood of λ.

Let r be such that $d_i(\lambda_0) < r$ for $i = 1,..., r$ and $d_i(\lambda_0) = \infty$ for $i = r + 1,..., n$. Define $\mu_1 = d_1(\lambda_0)$, $\mu_i = d_i(\lambda_0) - d_{i-1}(\lambda_0)$ for $i = 2,..., r$, and $\mu_i = \infty$ for $i = r + 1,..., n$. Then

$$0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n \leqslant \infty.$$

If $\mu_1 = \mu_2 = \cdots = \mu_n = 0$, then $M(\lambda)$ is said to be *without type exponents* at λ_0. Otherwise the *type exponents* of $M(\lambda)$ at λ_0 are $\mu_{k+1},..., \mu_n$, where μ_{k+1} is the first nonzero element of the sequence $\{\mu_1,..., \mu_n\}$.

Remark 3.1. The type exponents of $M(\lambda)$ are unchanged if $M(\lambda)$ is multiplied on the left or right by a matrix which is analytic and nonsingular at λ_0. In particular, one can apply to $M(\lambda)$ the elementary matrix operations of interchanging two rows (or columns), of adding to one row (or column) a multiple of another, or of multiplying a row (or column) by an analytic function different from zero at λ_0 without changing the type exponents. In this way $M(\lambda)$ can be reduced to a diagonal matrix from which one can, in theory, easily compute the type exponents [2, 3, 6].

Suppose now that $\lambda_0 \in \rho(A)$ and that $M(\lambda)$ is the perturbation matrix corresponding to (2.1). It follows from (2.4) and Remark 3.1 that $M(\lambda)$ and $W_1(\lambda)$ are equivalent at λ_0 in the sense that they have the same type exponents. In view of [2, Theorem II.6.1] we can therefore state:

THEOREM 3.1. *Let A be a bounded self-adjoint operator on the complex Hilbert space X and let B be a bounded finite-dimensional perturbation of A given by* (2.1). *Let $\lambda_0 \in \rho(A)$ and $W_1(\lambda) = \{\delta_{ij} + ((A - \lambda)^{-1} p_j, q_i)\}$ $(i, j = 1,..., n)$. Then:*

(i) *If $W_1(\lambda)$ is without type exponents at λ_0, then $\lambda_0 \in \rho(B)$.*

(ii) *If $W_1(\lambda)$ has type exponents $\mu_{k+1},..., \mu_n$ at λ_0, then all the type exponents are finite, and λ_0 is an isolated eigenvalue of B. Moreover, the generalized eigenspace V of B corresponding to λ_0 has a decomposition of the form*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_{n-k},$$

where each of the subspaces V_j has a basis of the form

$$\{v_j, (B - \lambda_0) v_j, ..., (B - \lambda_0)^{n_j - 1} v_j\}$$

with

$$(B - \lambda_0)^{n_j} v_j = 0 \quad \text{and} \quad n_j = \dim V_j = \mu_{k+j}, \quad j = 1,..., n - k.$$

Remark 3.2. The generalized eigenspace V is of course a reducing subspace for B. Briefly stated, Theorem 3.1 says that $W_1(\lambda)$ can be used to find the Jordan form of any matrix representation of B on V. Note that the algebraic multiplicity of λ_0 is $\dim V = \mu_{k+1} + \cdots + \mu_n = d_n(\lambda_0)$, while the geometric multiplicity is $n - k$, the nullity of $W_1(\lambda_0)$. Thus we recover the result of [7].

4. ISOLATED EIGENVALUES OF A; FORM OF THE PERTURBATION MATRIX

Theorem 3.1 allows us to investigate the eigenspace of B corresponding to any $\lambda_0 \in \mathscr{M}$ which is not an eigenvalue of A. In this section we consider the case when λ_0 is an eigenvalue of A. For convenience we make the following notational convention: when writing the components of a matrix, i and k will only be used as row indices; j and l will be used as column indices. Thus, e.g., $\{(u_l, q_i)\}$ $(i = 1,..., n,\ l = 1,..., \mu)$ denotes the $n \times \mu$ matrix with components $\gamma_{il} = (u_l, q_i)$ $(i = 1,..., n,\ l = 1,..., \mu)$.

THEOREM 4.1. *Let A, B, X be as in Theorem* 3.1, *and let λ_0 be an isolated eigenvalue of A of multiplicity μ. Let $u_1,..., u_\mu$ be an orthonormal basis for the eigenspace U of A corresponding to λ_0; let A' be the restriction of A to $U^\perp$; and let P' be the orthogonal projection of X onto $U^\perp$. Then the conclusions of Theorem* 3.1 *hold, but with $W_1(\lambda)$ replaced by the matrix*

$$W_2(\lambda) = \begin{bmatrix} \{\delta_{ij} + ((A' - \lambda)^{-1} P' p_j, q_i)\} & \{(u_l, q_i)\} \\ \{(p_j, u_k)\} & \{(\lambda - \lambda_0)\, \delta_{kl}\} \end{bmatrix} \quad \begin{bmatrix} i, j = 1,..., n \\ k, l = 1,..., \mu \end{bmatrix}. \tag{4.1}$$

Remark 4.1. $W_2(\lambda_0)$ is the "big" matrix defined in [7]. From Theorem 4.1 we recover Weinstein's result that the nullity of $W_2(\lambda_0)$ is the geometric multiplicity of λ_0 as an eigenvalue of B.

In order to prove Theorem 4.1 we use a technique from [2] to replace the given perturbation problem by one to which Theorem 3.1 applies. Specifically, we define a new operator $\hat{A} = A + (I - P')$, I being the identity operator. Then $\hat{A}$ is also a bounded self-adjoint operator, but $\lambda_0 \in \rho(\hat{A})$.

We next find a representation for B in terms of $\hat{A}$. Introduce a new basis

$$q_i^\circ = \sum_{j=1}^{n} \alpha_{ij} q_j \qquad (i = 1,..., n) \tag{4.2}$$

for the subspace S generated by $q_1 ,..., q_n$, where $q_1^\circ,..., q_n^\circ$ are so chosen that:

(i) $q_i' = P'q_i^\circ$ $(i = 1,..., m)$ forms a basis for the image $P'(S)$ of S under P' and $m = \dim P'(S)$.

(ii) $P'q_i^\circ = 0$ $(i = m + 1,..., n)$.

Clearly, such a basis exists. Then (2.1) can be replaced by

$$B = A + \sum_{i=1}^{n} (\cdot, q_i^\circ)\, p_i^\circ,$$

where $p_1^\circ,..., p_n^\circ$ satisfy

$$p_j = \sum_{i=1}^{n} p_j^\circ \bar{\alpha}_{ij} \qquad (j = 1,..., n). \tag{4.3}$$

Using the fact that $q_i^\circ = P'q_i^\circ + \sum_{k=1}^{\mu} (q_i^\circ, u_k)\, u_k$ $(i = 1,..., n)$ and $A = \hat{A} - \sum_{k=1}^{\mu}(\cdot, u_k)\, u_k$, we find that

$$B = \hat{A} + \sum_{i=1}^{m} (\cdot, q_i')\, p_i^\circ + \sum_{k=1}^{\mu} (\cdot, u_k)\,(v_k - u_k), \tag{4.4}$$

where

$$v_k = \sum_{i=1}^{n} (u_k , q_i^\circ)\, p_i^\circ \qquad (k = 1,..., \mu). \tag{4.5}$$

Equation (4.4) expresses B as a finite-dimensional perturbation of $\hat{A}$, with $q_1' ,..., q_m'$, $u_1 ,..., u_\mu$ linearly independent. Thus by Theorem 3.1, the structure of the generalized eigenspace of B corresponding to λ_0 can be found using the type exponents of the $(m + \mu) \times (m + \mu)$ matrix

$$\hat{W}_1(\lambda) = \begin{bmatrix} \{\delta_{ij} + ((\hat{A} - \lambda)^{-1} p_j^\circ, q_i')\} & \{((\hat{A} - \lambda)^{-1} (v_l - u_l), q_i')\} \\ \{((\hat{A} - \lambda)^{-1} p_j^\circ, u_k)\} & \{\delta_{kl} + ((\hat{A} - \lambda)^{-1} (v_l - u_l), u_k)\} \end{bmatrix}$$

$(i, j = 1,\ldots, m)$ $(k, l = 1,\ldots, \mu)$. We shall transform this matrix into the matrix $W_2(\lambda)$ of Theorem 4.1 using only operations which do not change the type exponents.

Since U reduces both A and $\hat{A}$, $((\hat{A} - \lambda)^{-1} p_j^\circ, q_i') = ((\hat{A} - \lambda)^{-1} p_j^\circ, P'q_i^\circ) = ((A' - \lambda)^{-1} P'p_j^\circ, q_i^\circ)$, and $((\hat{A} - \lambda)^{-1} (v_l - u_l), q_i') = ((\hat{A} - \lambda)^{-1} v_l, q_i') = ((A' - \lambda)^{-1} P'v_l, q_i^\circ)$ $(i, j = 1,\ldots, m)$ $(l = 1,\ldots, \mu)$. Since $\hat{A}$ is self-adjoint and $(\hat{A} - \lambda)^{-1} u_k = (1 + \lambda_0 - \lambda)^{-1} u_k$, also $((\hat{A} - \lambda)^{-1} p_j^\circ, u_k) = (1 + \lambda_0 - \lambda)^{-1} \times (p_j^\circ, u_k)$, and $\delta_{kl} + ((\hat{A} - \lambda)^{-1} (v_l - u_l), u_k) = (1 + \lambda_0 - \lambda)^{-1} ((v_l, u_k) - (\lambda - \lambda_0) \delta_{kl}$ $(j = 1,\ldots, m)$ $(k, l = 1,\ldots, \mu)$. Thus (see Remark 3.1) the type exponents of $\hat{W}_1(\lambda)$ at λ_0 are the same as those of

$$M_1(\lambda) = \begin{bmatrix} \{\delta_{ij} + ((A' - \lambda)^{-1} P'p_j^\circ, q_i^\circ)\} & \{-((A' - \lambda)^{-1} P'v_l, q_i^\circ)\} \\ \{(p_j, u_k)\} & \{(\lambda - \lambda_0) \delta_{kl} - (v_l, u_k)\} \end{bmatrix} \tag{4.6}$$

$(i, j = 1,\ldots, m)$ $(k, l = 1,\ldots, \mu)$.

Let $M_2(\lambda)$ be the $(n + \mu) \times (n + \mu)$ matrix defined by the right-hand side of (4.6) but with $(i, j = 1,\ldots, n)$. Since $((A' - \lambda)^{-1} P'p_j^\circ, q_i^\circ) = ((A' - \lambda)^{-1} P'v_l, q_i^\circ) = 0$ for $i = m + 1,\ldots, n$, we can, simply by rearranging rows (and columns) in $M_2(\lambda)$, transform it into the matrix

$$\hat{M}_2(\lambda) = \begin{bmatrix} M_1(\lambda) & N(\lambda) \\ 0 & I_{n-m} \end{bmatrix}.$$

Here I_{n-m} is the $(n - m) \times (n - m)$ identity matrix, and $N(\lambda)$ is an $m \times (n - m)$ matrix (whose exact form is not important). The type exponents of $M_2(\lambda)$ at λ_0 are therefore the same as those of $\hat{M}_2(\lambda)$, which are easily seen to be the same as those of $M_1(\lambda)$.

Next, using the definition (4.5) of $v_1,\ldots, v_\mu$, we see that

$$M_2(\lambda) = W_2^\circ(\lambda) \begin{bmatrix} I_n & \{-(u_l, q_i^\circ)\} \\ 0 & I_\mu \end{bmatrix} \quad \begin{pmatrix} i = 1,\ldots, n \\ l = 1,\ldots, \mu \end{pmatrix},$$

where

$$W_2^\circ(\lambda) = \begin{bmatrix} \{(\delta_{ij} + (A' - \lambda)^{-1} P'p_j^\circ, q_i^\circ)\} & \{(u_l, q_i^\circ)\} \\ \{(p_j^\circ, u_k)\} & \{(\lambda - \lambda_0) \delta_{kl}\} \end{bmatrix} \quad \begin{pmatrix} i, j = 1,\ldots, n \\ k, l = 1,\ldots, \mu \end{pmatrix}.$$

Finally, it follows easily from (4.2) and (4.3) that

$$W_2(\lambda) = Q^{-1} W_2^\circ(\lambda) Q,$$

where

$$Q = \begin{bmatrix} \{\tilde{\alpha}_{ij}\} & 0 \\ 0 & I_\mu \end{bmatrix} \quad (i, j = 1,\ldots, n).$$

It is now clear that $W_2(\lambda)$ has the same type exponents at λ_0 as $\hat{W}_1(\lambda)$, and the theorem is proved.

5. Examples

Suppose the bounded self-adjoint operator A has an isolated eigenvalue λ_0 of multiplicity four and corresponding orthonormal eigenvectors u_1, u_2, u_3, u_4. We consider two simple examples to which the results of the preceding sections can be applied.

Example 1. $B = A + (\cdot, u_1)\, u_2 + (\cdot, u_3)\, u_4$.

Here $W_1(\lambda) = I_2$. Thus $\det W_1(\lambda) \equiv 1$ and Aronszajn's rule gives the algebraic multiplicity $m_B(\lambda_0) = 4$. For λ near λ_0,

$$W_2(\lambda) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda - \lambda_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \lambda - \lambda_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - \lambda_0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \lambda - \lambda_0 \end{bmatrix}.$$

$W_2(\lambda_0)$ has nullity 2, so the geometric mulriplicity $\mu_B(\lambda) = 2$. $W_2(\lambda)$ is easily transformed by elementary matrix operations of the types described in Remark 3.1 into the matrix

$$\begin{bmatrix} I_4 & 0 \\ 0 & \begin{bmatrix} (\lambda - \lambda_0)^2 & 0 \\ 0 & (\lambda - \lambda_0)^2 \end{bmatrix} \end{bmatrix}$$

which has type exponents 2, 2 at λ_0. Thus the generalized eigenspace V of B corresponding to λ_0 decomposes into the direct sum of two irreducible reducing subspaces, each of dimension two. In fact, $V = V_1 \oplus V_2$, where $\{u_1, u_2 = (B - \lambda_0)\, u_1\}$ is a basis for V_1 and $\{u_3, u_4 = (B - \lambda_0)\, u_3\}$ is a basis for V_2.

Example 2. $B = A + (\cdot, u_1)\, u_2 + (\cdot, u_2)\, u_3$.

As in Example 1, $W_1(\lambda) = I_2$, $\det W_1(\lambda) \equiv 1$, and $m_B(\lambda_0) = 4$. Now, however,

$$W_2(\lambda) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda - \lambda_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \lambda - \lambda_0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda - \lambda_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda - \lambda_0 \end{bmatrix}.$$

Again, $\mu_B(\lambda_0) = 2$, but $W_2(\lambda)$ transforms into the matrix

$$\begin{bmatrix} I_4 & 0 \\ 0 & \begin{bmatrix} \lambda - \lambda_0 & 0 \\ 0 & (\lambda - \lambda_0)^3 \end{bmatrix} \end{bmatrix}$$

which has type exponents 1, 3. Hence the canonical decomposition of V is $V = V_1 \oplus V_2$, where $\dim V_1 = 1$ and $\dim V_2 = 3$. In fact, we can take $\{u_4\}$ as a basis for V_1 and $\{u_1, u_2 = (B - \lambda_0) u_1, u_3 = (B - \lambda_0)^2 u_1\}$ as a basis for V_2.

These simple examples show that although Aronszajn's rule and Weinstein's nullity rule enable us to find $m_B(\lambda_0)$ and $\mu_B(\lambda_0)$, the type exponents of the perturbation matrix may still be needed to find the structure of the generalized eigenspace.

6. Concluding Remarks

If A is not assumed to be self-adjoint, the simplifications of the present paper no longer apply, and the more general results of [2] must be used. However, Eq. (2.4) remains valid. Thus, although Eq. (2.5) and Theorem 3.1 must be replaced by [2, Theorem II.6.2] and [2, Theorem II.6.1], $W_1(\lambda)$ can still be used as the perturbation matrix corresponding to the representation (2.1).

The procedures used in the proof of Theorem 4.1 can also be adapted to the more general case, but the resulting perturbation matrices will be more complicated than is $W_2(\lambda)$.

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